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An improvement of the product integration method for a weakly singular Hammerstein equation

Laurence Grammont*, Hanane Kaboul †

Abstract

We present a new method to solve nonlinear Hammerstein equations with weakly singular kernels. The process to approximate the solution, followed usually, consists in adapting the discretization scheme from the linear case in order to obtain a nonlinear system in a finite dimensional space and solve it by any linearization method. In this paper, we propose to first linearize, via Newton method, the nonlinear operator equation and only then to discretize the obtained linear equations by the product integration method. We prove that the iterates, issued from our method, tends to the exact solution of the nonlinear Hammerstein equation when the number of Newton iterations tends to infinity, whatever the discretization parameter can be. This is not the case when the discretization is done first: in this case, the accuracy of the approximation is limited by the mesh size discretization. A Numerical example is given to confirm the theoretical result.

Keywords: Nonlinear equations, Newton-like methods, Product integration method, Integral equations.

AMS Classification: 65J15, 45G10, 35P05

1 Introduction

The general framework of this paper is the following. Let \mathcal{X} be a complex Banach space and $K : \mathcal{O} \subseteq \mathcal{X} \rightarrow \mathcal{X}$ a nonlinear Fréchet differentiable integral operator of the Hammerstein type defined on a nonempty open set \mathcal{O} of \mathcal{X} :

$$(1) \quad K(x)(s) := \int_a^b H(s, t) L(s, t) F(t, x(t)) dt, \quad \text{for all } x \in \Omega,$$

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where H is the singular part of the kernel, L is the regular part of the kernel and F , the nonlinear part of the operator, is a real-valued function of two variables :

$$(t, u) \in [a, b] \times \mathbb{R} \mapsto F(t, u) \in \mathbb{R},$$

with enough regularity so that K is twice Fréchet-differentiable on \mathcal{O} .

The problem is a nonlinear Fredholm integral equation of the second kind:

$$(2) \quad \text{Find } \varphi \in \mathcal{O} : \quad \varphi - K(\varphi) = y,$$

for a given function $y \in \mathcal{X}$.

Let $T := K'$ denote the Fréchet derivative of K , i.e., for all $x \in \mathcal{O}$,

$$(3) \quad T(x)h(s) = \int_a^b H(s, t)L(s, t) \frac{\partial F}{\partial u}(t, x(t))h(t) dt, \quad h \in \mathcal{X}, s \in [a, b].$$

In the following, \mathcal{X} will be the space of the real valued continuous functions over a real interval $[a, b]$, $C^0([a, b], \mathbb{R})$, equipped with the supremum norm $\|\cdot\|$.

If we consider a singular kernel such as $H(s, t) := \log(|s-t|)$ or $|s-t|^\alpha$, $0 < \alpha < 1$, an approximation based on standard numerical integrations is a poor idea. The main idea of the product integration method is introduced by Atkinson for linear integral equations ([7],[8] and [9]) and is motivated by Young [28]. The product integration method consists in performing a piecewise polynomial linear interpolation of the smooth part of the kernel times the function involving the unknown. This method is called product trapezoidal rule when the interpolation is linear. The solution of a second kind Fredholm integral equation with weakly singular kernel is typically nonsmooth near the boundary of the domain of integration. In order to obtain a high order of convergence, taking into account the singular behaviour of the exact solution, polynomial spline on graded mesh are developped (among other authors, Brunner, Peadar, Vainikko, Schneider [13], [24] and [26]). In [19], Kaneko, Noren and Xu discuss a standard product integration method with a general piecewise polynomial interpolation for weakly singular Hammerstein equation and indicate its superconvergence properties. Under particular assumptions on the right hand side y , on the function F defining the nonlinearity and on the regularity of the exact solution of (2), they give an error estimation involving n the discretization paramater and the degree m of the piecewise polynomial interpolation. Hence the error depends on n and m .

In the 1950's, the major theme in the domain of theoretical numerical analysis was the developement of general frameworks in the domain of functional analysis to build and analyze numerical methods. A particular important contribution in this context is the paper of Kantorovich [20] and later [21]. It proposes a generalization of Newton's method for solving nonlinear operator equations on Banach spaces. This idea is used everywhere when dealing with integral equations or partial differential equations. In [2], chapter 6, Anselone studies the Newton method to approximate solutions of nonlinear equations $P(x) = 0$ where P is a nonlinear differentiable operator from a Banach space into itself. When dealing with the convergence of approximate solutions, these are defined as the solution of $P_n(x) = 0$ where P_n is an approximation of P . The Newton method is then applied to the functional equation $P_n(x) = 0$. The

philosophy of most of the papers dealing with the numerical approximation of nonlinear integral operator equation consists in defining an approximate operator P_n to P and then apply Newton method ([4],[10], [11], [12], [13],[14],[16], [17],[19],[22],[23],[24] and [27]).

We propose to apply Newton's method directly to the operator equation $P(x) = 0$ and then to discretize the linear operator equations, issued from the Newton's iterations, by a product integration method. We will prove that the approximate iterates tend to the exact solution of the operator equation as the number of iterations tends to infinity. The important fact is that the convergence holds whatever the discretization parameter, defining the size of the linear system to be solved, can be. As we do not need the solution of each Newton iteration to be particularly accurate, we chose to apply the classical trapezoidal product integration method.

Section 2 is devoted to the description of our method (linearization via Newton's method followed by discretization by the product integration method). In Section 3, the convergence result is proved. In the last section, the classical method (discretization followed by linearization) is recalled and we compare it with our method through a numerical example.

2 Description of the new method

To solve the problem (2) for a given function y in $C^0([a, b], \mathbb{R})$, we propose to first apply the Newton method to the equation $\varphi - K(\varphi) = y$. It leads to the sequence $(\varphi^{(k)})_{k \geq 0} \in \mathcal{O}$:

$$(4) \quad \varphi^{(0)} \in \mathcal{O}, \quad (I - T(\varphi^{(k)}))(\varphi^{(k+1)} - \varphi^{(k)}) = -\varphi^{(k)} + K(\varphi^{(k)}) + y, \quad k \geq 0.$$

Then we discretize this equation with the product integration method associated to the piecewise linear interpolation.

Let Δ_n , defined by

$$(5) \quad a =: t_{n,0} < t_{n,1} < \dots < t_{n,n} := b,$$

be the uniform grid of $[a, b]$ with mesh $h_n := \frac{b-a}{n}$.

Setting

$$f_x(t) := \frac{\partial F}{\partial u}(t, x(t)),$$

$[L(s, t)f_x(t)h(t)]_n$ denotes the piecewise linear interpolation of $L(s, t)f_x(t)h(t) :$
 $\forall s \in [a, b], \forall i = 1, \dots, n$:

$$\begin{aligned} [L(s, t)f_x(t)h(t)]_n &:= \frac{1}{h_n}(t_{n,i} - t)L(s, t_{n,i-1})f_x(t_{n,i-1})h(t_{n,i-1}) \\ &\quad + \frac{1}{h_n}(t - t_{n,i-1})L(s, t_{n,i})f_x(t_{n,i})h(t_{n,i}) \end{aligned}$$

for $t \in [t_{n,i-1}, t_{n,i}]$.

We define the approximate operator T_n by

$$(6) \quad T_n(x)(h)(s) := \int_a^b H(s, t) [L(s, t) f_x(t) h(t)]_n dt, \quad x \in \mathcal{O}, h \in \mathcal{X}, s \in [a, b].$$

Then the approximate problem is:

$$(7) \quad \text{Find } \varphi_n^{(k+1)} \in X : (I - T_n(\varphi_n^{(k)}))(\varphi_n^{(k+1)} - \varphi_n^{(k)}) = -\varphi_n^{(k)} + K(\varphi_n^{(k)}) + y.$$

We have

$$\begin{aligned} T_n(x)(h)(s) &:= \sum_{j=0}^n w_{n,j}(s) L(s, t_{n,j}) f_x(t_{n,j}) h(t_{n,j}), \\ w_{n,0}(s) &:= \frac{1}{h_n} \int_{t_{n,0}}^{t_{n,1}} H(s, t) (t_{n,1} - t) dt, \\ w_{n,n}(s) &:= \frac{1}{h_n} \int_{t_{n,n-1}}^{t_{n,n}} H(s, t) (t - t_{n,n-1}) dt, \end{aligned}$$

and for $1 \leq j \leq n-1$,

$$w_{n,j}(s) := \frac{1}{h_n} \int_{t_{n,j-1}}^{t_{n,j}} H(s, t) (t - t_{n,j-1}) dt + \frac{1}{h_n} \int_{t_{n,j}}^{t_{n,j+1}} H(s, t) (t_{n,j+1} - t) dt.$$

Hence (7) can be rewritten as

$$(8) \quad \varphi_n^{(k+1)}(s) - \sum_{j=0}^n w_{n,j}(s) L(s, t_{n,j}) f_k(t_{n,j}) \varphi_n^{(k+1)}(t_{n,j})$$

$$(9) \quad = K(\varphi_n^{(k)})(s) + y(s) - \sum_{j=0}^n w_{n,j}(s) L(s, t_{n,j}) f_k(t_{n,j}) \varphi_n^{(k)}(t_{n,j}),$$

where

$$f_k(t_{n,j}) := \frac{\partial F}{\partial u}(t_{n,j}, \varphi_n^{(k)}(t_{n,j})).$$

Setting

$$x_n^{(k+1)}(j) := \varphi_n^{(k+1)}(t_{n,j}),$$

From the evaluations of equation (26) at the nodes of the grid, it is straightforward that the vector $x_n^{(k+1)}$ is the solution of the linear system

$$(10) \quad (I - A_n^{(k)}) x_n^{(k+1)} = b_n^{(k)},$$

where

$$\begin{aligned} A_n^{(k)}(i, j) &:= w_{n,j}(t_{n,i})L(t_{n,i}, t_{n,j})f_k(t_{n,j}), \\ b_n^{(k)} &:= K(\varphi_n^{(k)})(t_{n,i}) + y(t_{n,i}) - A_n^{(k)}x_n^{(k)}. \end{aligned}$$

$\varphi_n^{(k+1)}$ is recovered from equation (26) :

$$\begin{aligned} \varphi_n^{(k+1)}(s) &= \sum_{j=1}^n w_{n,j}(s)L(s, t_{n,j})f_k(t_{n,j}) \left(x_n^{(k+1)}(t_{n,j}) - x_n^{(k)}(t_{n,j}) \right) \\ &+ K(\varphi_n^{(k)})(s) + y(s). \end{aligned}$$

3 Convergence property of the new method

Existence, uniqueness and regularity properties of the solution of equation (2) have been already considered (for example by Kaneko, Noren and Xu [18] or Pedas and Vainikko [25]). In this section, we are only interested by the proof of the convergence of $\varphi_n^{(k)}$ towards φ when $k \rightarrow \infty$.

Hypotheses:

(H0) F , defined in (1), is twice continuously differentiable on $[a, b] \times \mathbb{R}$.

(H1) $L \in C^0([a, b] \times [a, b], \mathbb{R})$.

(H2) H verifies:

(H2.1) $c_H := \sup_{s \in [a, b]} \int_a^b |H(s, t)| dt < +\infty$.

(H2.2) $\lim_{h \rightarrow 0} \omega_H(h) = 0$, where

$$\omega_H(h) := \sup_{|s-\tau| \leq h, s, \tau \in [a, b]} \int_a^b |H(s, t) - H(\tau, t)| dt.$$

(H3) $\varphi \in \mathcal{O}$ is an isolated solution of $\varphi - K(\varphi) = f$.

(H4) $I - T(\varphi)$ is invertible.

These assumptions ensure that equations (2),(7) and (26) for n large enough, are uniquely solvable (see [9]).

Let $a > 0$ such that $B(\varphi, a) \subset \mathcal{O}$, where $B(\varphi, a)$ denotes the open ball in $C^0([a, b], \mathbb{R})$ centered in φ and of radius a . As $\varphi \in C^0([a, b], \mathbb{R})$, φ is bounded. As F is twice continuously differentiable, the following constant exists:

$$M_2(a) := \sup_{t \in [a, b], |u| \leq a + \|\varphi\|} \left| \frac{\partial^2 F}{\partial u^2}(t, u) \right|.$$

The proof of convergence relies on the successive approximations convergence result (see [23] Theorem 2.3. pp 21). Let us recall this result (in a form slicely different from [23]).

Proposition 1 *Consider a nonlinear operator A from a Banach space \mathcal{X} into itself, defined on an open set \mathcal{V} . Let $x^* \in \mathcal{V}$ be a fixed point of A . Let the operator A be Fréchet differentiable at the point x^* . Let us assume that the following condition is fulfilled*

$$(11) \quad \rho_0 := \rho(A'(x^*)) < 1,$$

where ρ denotes the spectral radius and A' denotes the Fréchet derivative of A .

Then, for all $\varepsilon > 0$ such that $\rho_0 + \varepsilon < 1$, there exist $r_\varepsilon > 0$ and $r'_\varepsilon > 0$ such that $B(x^*, r_\varepsilon) \subset \mathcal{V}$ and $B(x^*, r'_\varepsilon) \subset \mathcal{V}$ and such that for $x^{(0)}$ in $B(x^*, r'_\varepsilon)$, the successive approximations $x^{(k)}$ defined by

$$x^{(k+1)} = A(x^{(k)}), \quad k \in \mathbb{N},$$

remain in $B(x^*, r_\varepsilon)$ for all $k \in \mathbb{N}$, and the sequence $(x^{(k)})_{k \geq 0}$ converges to x^* . Moreover,

$$\|x^{(k)} - x^*\| \leq r_\varepsilon(\rho_0 + \varepsilon)^k.$$

The following four lemmas are needed to prove our main result.

Lemma 1 *For all $a > 0$ such that $B(\varphi, a) \subset \mathcal{O}$, for all $x \in B(\varphi, a)$,*

$$(12) \quad \|T(x) - T(\varphi)\| \leq c_H c_L M_2(a) \|x - \varphi\|,$$

where

$$c_L := \max_{s, t \in [a, b]} |L(s, t)|.$$

Proof : From the mean value theorem applied to $\frac{\partial F}{\partial u}$, for all t , there exist a real number $c(t)$ between $\varphi(t)$ and $x(t)$ such that

$$\frac{\partial F}{\partial u}(t, x(t)) - \frac{\partial F}{\partial u}(t, \varphi(t)) = \frac{\partial^2 F}{\partial u^2}(t, c(t))(x(t) - \varphi(t)).$$

As $\forall t \in [a, b]$, $|c(t)| \leq r + \|\varphi\|$,

$$(13) \quad \|T(x) - T(\varphi)\| \leq \|x - \varphi\| \sup_{s \in [a, b]} \int_a^b |H(s, t)| |L(s, t)| \left| \frac{\partial^2 F}{\partial u^2}(t, c(t)) \right| dt$$

$$(14) \quad \leq c_H c_L M_2(a) \|x - \varphi\|.$$

Hence (13) is deduced. ■

Now a is fixed such that $B(\varphi, a) \subset \mathcal{O}$.

Lemma 2 *There is a positive number $r < a$ such that for all $x \in B(\varphi, r)$, $I - T(x)$ is invertible and*

$$(15) \quad \|(I - T(x))^{-1}\| \leq 2\mu,$$

where $\mu := \|(I - T(\varphi))^{-1}\|$.

Moreover, for all $x \in B(\varphi, r)$, for n large enough, $I - T_n(x)$ is invertible and there exists a constant $c_x > 0$, independent of n , such that

$$(16) \quad \|(I - T_n(x))^{-1}\| \leq c_x.$$

Proof :

Let $0 < r < a$ be such that $r \leq \frac{1}{2\mu c_H c_L M_2(a)}$.

For all $x \in B(\varphi, r)$,

$$I - T(x) = I - T(\varphi) + T(\varphi) - T(x) = (I - T(\varphi))[I + (I - T(\varphi))^{-1}(T(\varphi) - T(x))].$$

Since

$$\|(I - T(\varphi))^{-1}(T(\varphi) - T(x))\| \leq \mu \|T(\varphi) - T(x)\| \leq \mu c_H c_L M_2(a) r \leq \frac{1}{2},$$

we conclude that $I - T(x)$ is invertible and that its inverse is uniformly bounded on $B(\varphi, r)$. In fact

$$(I - T(x))^{-1} = [I + (I - T(\varphi))^{-1}(T(\varphi) - T(x))]^{-1}(I - T(\varphi))^{-1},$$

so

$$\|(I - T(x))^{-1}\| \leq \mu \sum_{k=0}^{\infty} \|(I - T(\varphi))^{-1}(T(\varphi) - T(x))\|^k \leq 2\mu.$$

The function $(s, t) \rightarrow L(s, t) \frac{\partial F}{\partial u}(t, x(t))$ is in $C^0([a, b] \times [a, b], \mathbb{R})$. Hence, according to [9] or [2], $T_n(x) \xrightarrow{p} T(x)$, where \xrightarrow{p} denotes the pointwise convergence, and the sequence $(T_n(x))_{n \geq 0}$ is collectively compact.

For all $x \in B(\varphi, r)$, $T_n(x)$ is a collectively compact approximation of $T(x)$ (see [2]). Hence for n large enough, $I - T_n(x)$ is invertible and $(I - T_n(x))^{-1}$ is uniformly bounded in n . This means that there is a constant c_x such that for n large enough,

$$\|(I - T_n(x))^{-1}\| \leq c_x.$$

This ends the proof. ■

Let A_n be the operator defined on $B(\varphi, r)$ by

$$(17) \quad A_n(x) := x + S_n(x) (K(x) + y - x),$$

where

$$(18) \quad S_n(x) := (I - T_n(x))^{-1}.$$

Notice that we have

$$(19) \quad A_n(\varphi) = \varphi.$$

Lemma 3 *The operator S_n is Fréchet differentiable at φ .*

Proof: As for all $h \in \mathcal{X}$, $T_n(x)(h)(s) = \sum_{j=0}^n w_{n,j}(s) L(s, t_{n,j}) \frac{\partial F}{\partial u}(t_{n,j}, x(t_{n,j})) h(t_{n,j})$, the operator $x \mapsto T_n(x)$ is differentiable at φ , and we have for all $h \in \mathcal{X}$ and $\delta \in \mathcal{X}$,

$$T'_n(x)(\delta, h)(s) = \sum_{j=0}^n w_{n,j}(s) L(s, t_{n,j}) \frac{\partial^2 F}{\partial u^2}(t_{n,j}, x(t_{n,j})) \delta(t_{n,j}) h(t_{n,j}).$$

As, at the first order, $(L + E)^{-1} - (L)^{-1} \simeq -(L)^{-1} E (L)^{-1}$, we have

$$\begin{aligned} S_n(\varphi + \delta) - S_n(\varphi) &= (I - T_n(\varphi + \delta))^{-1} - (I - T_n(\varphi))^{-1} \\ &= (I - T_n(\varphi) - T'_n(\varphi)\delta + O(\delta^2))^{-1} - (I - T_n(\varphi))^{-1} \\ &\simeq S_n(\varphi) T'_n(\varphi) \delta S_n(\varphi), \end{aligned}$$

so that S_n is Fréchet differentiable at φ and

$$S'_n(\varphi)\delta = S_n(\varphi) T'_n(\varphi) \delta S_n(\varphi).$$

■

Lemma 4 *The operator A_n is Fréchet differentiable at φ . For n large enough,*

$$(20) \quad \rho(A'_n(\varphi)) < 1.$$

Proof: Notice that $A_n(\varphi) = \varphi$, hence

$$\begin{aligned} A_n(\varphi + h) - A_n(\varphi) &= \varphi + h + S_n(\varphi + h) (K(\varphi + h) + y - \varphi - h) - \varphi \\ &= h + S_n(\varphi + h) (K(\varphi) + T(\varphi)h + O(h^2) + y - \varphi - h) \\ &= h + S_n(\varphi + h) ((T(\varphi) - I)h + O(h^2)), \end{aligned}$$

hence A_n is differentiable at φ and

$$(21) \quad A'_n(\varphi) = I - S_n(\varphi)(I - T(\varphi)).$$

We have

$$(22) \quad \rho(I - S_n(\varphi)(I - T(\varphi))) = \inf_n \|(I - S_n(\varphi)(I - T(\varphi)))^n\|^{\frac{1}{n}}$$

$$(23) \quad \leq \|(I - S_n(\varphi)(I - T(\varphi)))^2\|^{\frac{1}{2}}.$$

Since $(I - S_n(\varphi)(I - T(\varphi))) = S_n(\varphi)(T(\varphi) - T_n(\varphi))$,

$$\begin{aligned} \|(I - S_n(\varphi)(I - T(\varphi)))^2\| &= \|S_n(\varphi)(T(\varphi) - T_n(\varphi))S_n(\varphi)(T(\varphi) - T_n(\varphi))\| \\ &\leq \|S_n(\varphi)\| \|(T(\varphi) - T_n(\varphi))S_n(\varphi)(T(\varphi) - T_n(\varphi))\| \\ &\leq c_\varphi \|(T(\varphi) - T_n(\varphi))S_n(\varphi)(T(\varphi) - T_n(\varphi))\|. \end{aligned}$$

As $S_n(\varphi)$ is uniformly bounded (see Lemma 2) and $(T(\varphi) - T_n(\varphi))$ is collectively compact, the closure of the set $S := \cup_n \{S_n(\varphi)(T(\varphi) - T_n(\varphi))x, \|x\| \leq 1\}$ is compact so that

$$\|(I - S_n(\varphi)(I - T(\varphi)))^2\| \leq c_\varphi \sup_{x \in S} \|(T(\varphi) - T_n(\varphi))x\| \xrightarrow{n \rightarrow \infty} 0.$$

Then

$$\rho(A'_n(\varphi)) \xrightarrow{n \rightarrow \infty} 0.$$

This ends the proof. ■

Theorem 1 *Under assumptions (H0) to (H4), there exists $r > 0$ such that, for a fixed n large enough to have*

$$\rho_n := \rho(A'_n(\varphi)) < 1,$$

and for any $\varepsilon > 0$ such that $\rho_n + \varepsilon < 1$, there exist $B(\varphi, r'_{n,\varepsilon}) \subset B(\varphi, r)$ and $B(\varphi, r_{n,\varepsilon}) \subset B(\varphi, r)$ such that, if $\varphi_n^{(0)} \in B(\varphi, r'_{n,\varepsilon})$, then the sequence $(\varphi_n^{(k)})_{k \geq 0}$ solution of

$$(I - T_n(\varphi_n^{(k)}))(\varphi_n^{(k+1)} - \varphi_n^{(k)}) = -\varphi_n^{(k)} + K(\varphi_n^{(k)}) + f,$$

is defined, belongs to $B(\varphi, r_{n,\varepsilon})$ and

$$\varphi_n^{(k)} \xrightarrow{k \rightarrow \infty} \varphi.$$

Moreover, the following estimation holds:

$$(24) \quad \|\varphi_n^{(k)} - \varphi\| \leq r_{n,\varepsilon} (\rho_n + \varepsilon)^k.$$

Proof : The exact solution φ of the nonlinear problem (2) is a fixed point of A_n (see (19)). From Lemma 4, A_n is Fréchet differentiable at φ and for n large enough $\rho(A'_n(\varphi)) < 1$. The conditions needed to apply Proposition 1 to the operator A_n are satisfied so that Proposition 1 gives the result. ■

4 Numerical evidence

4.1 The classical product integration method

Let us recall the classical product integration method applied to nonlinear operators (see[19]). The classical product integration approximation ψ_n solves the nonlinear equation

$$\psi_n(s) - \int_a^b H(s, t)[L(s, t)F(t, \psi_n(t))]_n dt = y(s),$$

where $[L(s, t)F(t, \psi_n(t))]_n$ denotes the piecewise linear interpolant of $t \mapsto L(s, t)F(t, \psi_n(t))$. Using the uniform grid Δ_n , we obtain

$$(25) \quad \psi_n(s) - \sum_{j=0}^n w_{n,j}(s)L(s, t_{n,j})F(t_{n,j}, \psi_n(t_{n,j})) = y(s),$$

where

$$\begin{aligned} w_{n,0}(s) &:= \frac{1}{h_n} \int_{t_{n,0}}^{t_{n,1}} H(s, t)(t_{n,1} - t)dt, \\ w_{n,j}(s) &:= \frac{1}{h_n} \int_{t_{n,j-1}}^{t_{n,j}} H(s, t)(t - t_{n,j-1})dt + \frac{1}{h_n} \int_{t_{n,j}}^{t_{n,j+1}} H(s, t)(t_{n,j+1} - t)dt, \quad j = 1, \dots, n-1, \\ w_{n,n}(s) &:= \frac{1}{h_n} \int_{t_{n,n-1}}^{t_{n,n}} H(s, t)(t - t_{n,n-1})dt. \end{aligned}$$

Set

$$\begin{aligned} Y_n(i) &:= y(t_{n,i}), \\ A_n(i, j) &:= w_{n,j}(t_{n,i})L(t_{n,i}, t_{n,j}), \\ X_n &:= \begin{bmatrix} \psi_n(t_{n,0}) \\ \vdots \\ \psi_n(t_{n,n}) \end{bmatrix}, \\ F(X_n) &:= \begin{bmatrix} F(t_{n,0}, \psi_n(t_{n,0})) \\ \vdots \\ F(t_{n,n}, \psi_n(t_{n,n})) \end{bmatrix}. \end{aligned}$$

Evaluating the equation (25) at the nodes $t_{n,i}$, $i = 0, \dots, n$, the following non linear system is obtained:

$$(26) \quad X_n - A_n F(X_n) = Y_n,$$

which we can solve by the classical finite dimensional Newton method ([5] and [6]). Notice that the Newton's iterates tends to ψ_n when $k \rightarrow \infty$ and not to φ , the exact solution. It means that the accuracy of the approximate solution is limited by n .

4.2 Numerical Illustration

Numerical experiments are now carried out to illustrate the accuracy of our method. Let us consider in $\mathcal{X} := \mathbb{C}^0([0, 1], \mathbb{R})$, the operator

$$K(\varphi)(s) := \int_0^1 \kappa(s, t, \varphi(t)) dt, \quad \varphi \in \mathcal{O} \subseteq \mathcal{X}, \quad s \in [0, 1],$$

with the real valued kernel function κ :

$$(s, t, u) \in [0, 1] \times [0, 1] \times \mathbb{R} \mapsto \kappa(s, t, u) := \log(|s - t|) \sin(\pi u).$$

The exact solution of $\varphi(s) - K(\varphi)(s) = y(s)$ is

$$\varphi(s) := 1$$

for

$$y(s) := 1.$$

Implementation remark : To solve the linear system at each Newton iteration, the integral $\int_a^b \log(|s - t|) \sin(\varphi_n^{(k)}(t)) dt$ needs to be evaluated. To evaluate it, we use the singularity subtraction technique ([3] and also [1]).

In the following, our method is called the linearization-discretization method and the classical one is called the discretization-linearization method. We compare them.

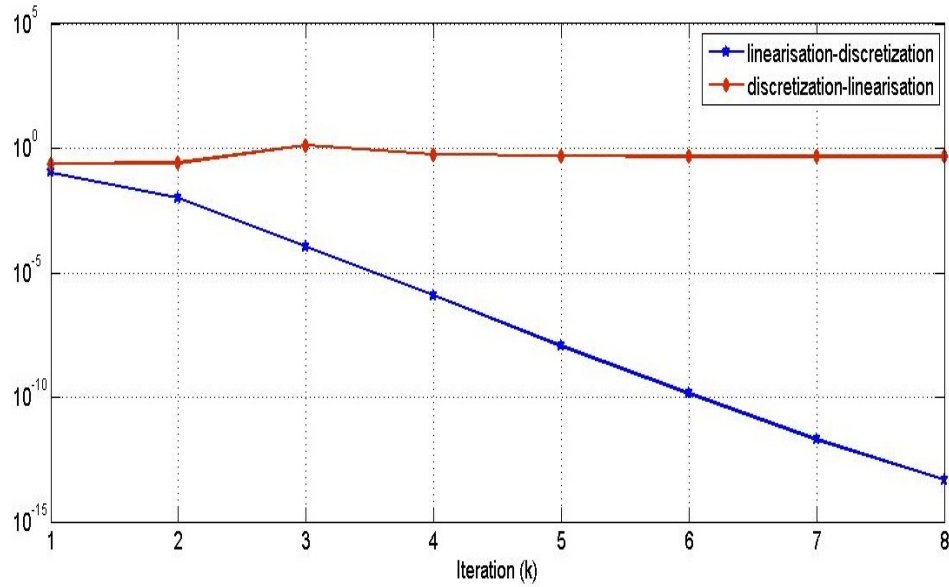


Figure 1: Comparison of the errors for $n = 50$

The classical discretization-linearization method can not be accurate if it is performed on a coarse grid ($n = 50$). We obtain the approximate solution ψ_n of φ and the error is constant with the number of iterations. On the other hand, our linearization-discretization method approaches the exact solution φ even if the discretization is done with a coarse grid. It confirms the theoretical result.

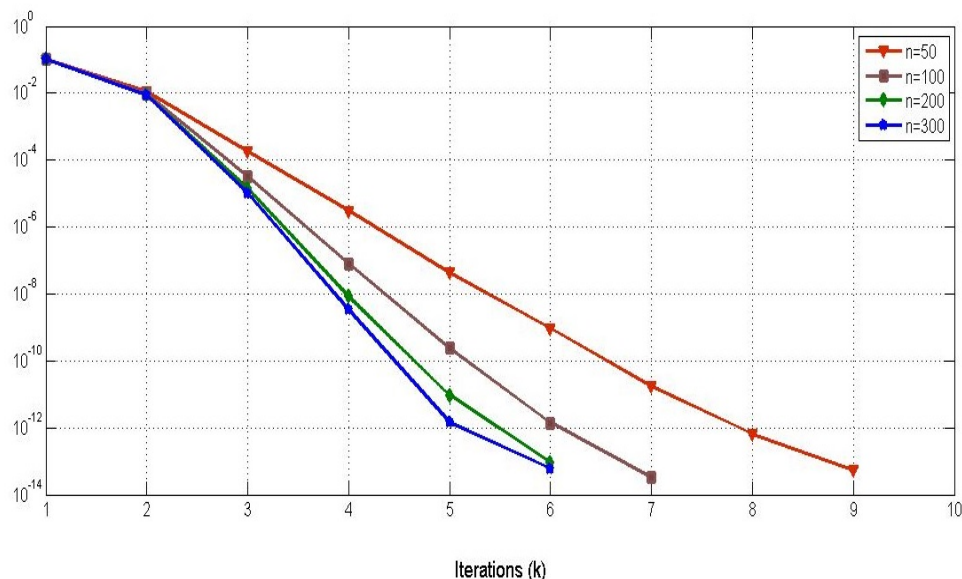


Figure 2: errors of the linearization-discretization method for increasing values of n

When n increases, the number of Newton's iterations needed to reach a fixed accuracy decreases. It can be useful to apply higher order product integration methods (see the idea of Diogo, Franco and Lima in [15]), instead of the product trapezoidal rule, to reduce the number of Newton's iterations and therefore the computational cost of the method. An high order product integration approximation, adapted to a Fredholm equation, can also be a good starting point for the Newton's iterations of our method.

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